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Journal of Algebra 320 (2008) 2559–2566
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Pullbacks of Prüfer rings

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Received 12 February 2008
Available online 7 July 2008
Communicated by Luchezar L. Avramov

Abstract

In this paper we consider five extensions of the Prüfer domain notion to commutative rings with zero-divisors and investigate their behavior in a special type of pullback called a conductor square. That is, for a pair of rings $R \subset T$ with non-zero conductor of T into R, we find necessary and sufficient conditions on the rings T, T/C, and R/C in order that R has one of the five Prüfer conditions.

Keywords: Pullback; Prüfer domain; Zero-division

1. Introduction

As noted by Gilmer in [10], Prüfer domains play a central role in non-Noetherian commutative ring theory. Since the introduction of the concept in 1932, much progress has been made in the development of their theory and there are now a myriad of equivalent definitions for a Prüfer domain. For example, one might define a Prüfer domain as the non-Noetherian version of a Dedekind domain or as the global version of a valuation domain. We refer the reader to [1,10], or [14] for a more complete list of equivalent conditions.

It has become fashionable in recent years to study integral domains (especially Prüfer domains) via pullback diagrams. It is well worth noting that pullback constructions provide a rich source of examples and counter examples in commutative algebra (see [18]). Of particular interest is a special type of pullback diagram called a *conductor square*. Let R and T be any commutative rings with $R \subset T$, and suppose that R and T have a common, non-zero ideal. We call the largest, non-zero, common ideal C the *conductor* of T into R. Setting $A = R/C$

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.06.005

and $B = T/C$, we obtain the natural surjections $\eta_1 : T \twoheadrightarrow B$ and $\eta_2 : R \twoheadrightarrow A$ and the inclusions $\iota_1 : A \hookrightarrow B$ and $\iota_2 : R \hookrightarrow T$. These maps yield a commutative diagram, called a conductor square, which defines R as a pullback of η_1 and ι_1,

$$\begin{array}{ccc} R & \hookrightarrow & T \\ \downarrow & & \downarrow \\ A & \hookrightarrow & B. \end{array} \tag{$\square$}$$

If we start with a ring surjection $\eta_1 : T \twoheadrightarrow B$ and an inclusion of rings $\iota_1 : A \hookrightarrow B$, then the pullback defines a subring R of T with conductor $C = \ker \eta_1$ and conductor square ($\square$). For additional information on pullbacks, we refer the reader to [7, Chapter 1].

It is common in the study of pullback constructions to assume that T is an integral domain and that C is a maximal ideal of T. Many authors have investigated various ring and ideal-theoretic properties that transfer in this type of diagram. For example, [5,8,9], and [20] are all devoted to the conductor square ($\square$), where the rings T, R and A are integral domains and B is a field.

However, interesting examples can be obtained by allowing zero-divisors in the pullback square. For example, let D be an integral domain with field of fractions K and let $E = \{e_1, \ldots, e_r\}$ be any finite subset of D. Setting $C = (x - e_1) \cdot \ldots \cdot (x - e_r)K[X]$ and $A = \prod_{i=1}^{r} D$, we get $R = \mathrm{Int}(E, D) = \{g \in K[X]\colon\ g(E) \subset D\}$, the ring of integer-valued polynomials on D determined by the subset E, is defined by the conductor square ($\square$), where the rings A and B are not integral domains (see [19, Proposition 5] and [3, Examples 4.4]). Thus, a natural question arises at this point. What ring and ideal-theoretic properties transfer in the conductor square when the conductor is not a maximal (or even a prime) ideal of T? (See [6, Open Problem (50)].) If E is finite, it is known that $\mathrm{Int}(E, D)$ is a Prüfer domain if and only if D is a Prüfer domain (see [19, Corollary 7]). Also, for $n \geqslant 2$, the ring $\mathrm{Int}(E, D)$ has the n-generator property for finitely generated ideals if and only if D has the same property [4, Corollary 4].

As for the transfer of other ring theoretic properties in the more general setup of ($\square$), some progress has been made. In [11, Theorem 5.1.3], it is shown that if C is a flat ideal of T, then R is a coherent ring if and only if A is a Noetherian ring and T is a coherent ring. In [3, Theorem 3.3], it is shown that R is an arithmetical ring if and only if A and T are arithmetical rings and B is locally an overring of A at every prime ideal of R (see below for definitions). In [15, Theorem 2.1] it is shown that under certain conditions, universal catenarity behaves nicely in a conductor square.

In this article we consider five extensions of the Prüfer domain notion to commutative rings with zero-divisors and investigate their behavior in the conductor square ($\square$). We make the following definitions:

Definition 1.1.

(1) We call a ring R *semi-hereditary* if every finitely generated ideal of R is projective.
(2) We say that R has *weak global dimension* $\leqslant 1$ (wk.gl.dim.$(R) \leqslant 1$) if every finitely generated ideal of R is flat.
(3) We call a ring R an *arithmetical ring* if the lattice formed by its ideals is distributive.
(4) We call a ring R a *Gaussian ring* if for every $f, g \in R[X]$, one has the content ideal equation $c(fg) = c(f)c(g)$.
(5) We call a ring R a *Prüfer ring* if every finitely generated regular ideal is invertible.

Although these conditions are equivalent for Prüfer domains, for commutative rings in general it is shown in [12] and [13] that one has the strict implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). In [2] exact conditions for reversing the implication arrows are found by imposing extra conditions on the total ring of quotients of R. For example, it is shown that, for $n = 1, 2, 3, 4$, a ring R has property (n) if and only if its total ring of quotients $Q(R)$ has property (n) and R has property $(n + 1)$.

The content of this paper is organized as follows. In Section 2, we recall some basic facts about pullback diagrams. In Section 3, we further investigate the five Prüfer conditions focusing on localizations and overrings. In Section 4, we prove our main result (Theorem 4.2): For the conductor square ($\Box$), we find necessary and sufficient conditions on A, T, and B in order that R satisfies condition (n) for $n = 1, 2, 3, 4, 5$.

2. Local rings and pullbacks

We begin with some terminology and fix notation that will be used in the sequel. We call an element of the ring R a *regular* element if it is not a zero-divisor in R and we call an ideal of R a *regular* ideal if it contains a regular element. We denote by $Z(R)$ the set of all zero-divisors of R. If $S_0 = R - Z(R)$, then the localization $S_0^{-1}R$ is called the *total quotient ring* of R, which we shall denote by $Q(R)$. A ring T is called an *overring* of R if $R \hookrightarrow T \hookrightarrow Q(R)$. We will call the diagram ($\Box$) a *regular conductor square* if the ideal C is a regular ideal.

In this section we recall several properties that hold in any conductor square. The statements of these results may be found in [7, Chapter 1]. We provide proofs so that the general mechanics of the conductor square are illustrated.

Lemma 2.1. *Consider the regular conductor square* ($\Box$)*.*

(1) *T is an overring of R.*
(2) *If $T \simeq S^{-1}R$ for some multiplicatively closed set $S \subset R$, then $B \simeq S^{-1}A$. Moreover, B is an overring of A.*
(3) *If R is a local ring then there is a* 1–1 *correspondence between the maximal ideals of B and the maximal ideals of T.*
(4) *If $P \in \mathrm{Spec}(R)$ and $C \nsubseteq P$, then there is a unique $Q \in \mathrm{Spec}(T)$ such that $R_P \simeq T_Q$ where $Q \cap R = P$.*
(5) *If A and T are local rings, then R is a local ring.*

Proof. (1) Choose any $t \in T$ and any regular element $c \in C$. One easily checks that the map $T \to Q(R) : t \mapsto \frac{ct}{c}$ is well-defined ($ct, c \in C \leqslant R$) and injective. It follows that $R \hookrightarrow T \hookrightarrow Q(R)$.

(2) If $T \simeq S^{-1}R$, then $S^{-1}A \simeq S^{-1}(R/C) \simeq (S^{-1}R)/(S^{-1}C) \simeq T/C = B$. If there were a zero-divisor in S, then $sa = 0$ for some non-zero element $a \in A$. This implies that $\frac{a}{1} = 0$ in $B \simeq S^{-1}A$. But this contradicts the fact that $A \hookrightarrow B$.

(3) It suffices to show that every maximal ideal of T contains C. Chose any $c \in C$ and any $t \in T$. Then $ct \in C \leqslant R$ so that $1 - ct$ is a unit in R since R is local. Thus $1 - ct$ is a unit in T for every $t \in T$. It follows that c belongs to the Jacobson radical of T, so that C is contained in every maximal ideal of T.

(4) Since $C \nsubseteq P$, we may choose an element $c \in C - P$. Localizing at the monoid generated by c, we obtain the isomorphism of rings $R_c \simeq T_c$. The equation $\frac{t}{c^k} = \frac{tc}{c^{k+1}}$ ensures surjectivity

since $tc \in R$. Now, P survives in R_c so there is a unique prime ideal QT_c that corresponds to the prime ideal PR_c. Thus, we have the canonical isomorphism $R_P \simeq (R_c)_{P_c} \simeq (T_c)_{Q_c} \simeq T_Q$. One easily checks that $Q \cap R = P$.

(5) Let $\overline{M}$ be the unique maximal ideal of A and suppose that $r \in R - M$. Then $\bar{r}$ is a unit in A and hence, $\bar{r}^{-1} \in A$. It follows that $\bar{r}$ is a unit in B, so that $\bar{r} \notin \overline{N}$, the unique maximal ideal of B. We now have that $r \notin N$, the unique maximal ideal of T. This means that r is a unit of T, so that $r^{-1} \in T$. But $\overline{r^{-1}} = \bar{r}^{-1} \in A$ proves that $r^{-1} \in R$ and that r is a unit of R. It follows that M is the unique maximal ideal of R. □

We close this section by noting that the regularity of the ideal C is only required in (1) of Lemma 2.1.

3. Prüfer conditions in rings with zero-divisors

In this section we further explore rings with Prüfer conditions by considering their overrings and localizations. First, we need to fix some more terminology. We call a ring R a *Von Neuman regular ring* (VNR) if for every $a \in R$, there exists $b \in R$ such that $a^2b = a$. We call an integral domain D a *valuation domain* if its ideals are totally ordered. We call a ring R a *chained ring* if its ideals are totally ordered. Thus, a chained ring with no zero-divisors is a valuation domain. It is useful to have alternative characterizations of the five Prüfer conditions at our disposal. First, we recall a well-known fact relating projectivity and invertibility of finitely generated regular ideals.

Lemma 3.1. *Let R be a commutative ring and let $I = (a_1, \dots, a_n)$ be any finitely generated regular ideal of R. Then the following statements are equivalent*:

(1) *I is an invertible ideal,*
(2) *I is a projective R-module,*
(3) *For each prime ideal P of R, there is $i \leqslant n$ such that $a_i R_P = I R_P$.*

We now summarize the relationship between a commutative ring R with Prüfer condition (n) and its localizations R_P at prime (maximal) ideals.

Theorem 3.2. *Let R be a commutative ring.*

(1) [12] *R is semi-hereditary if and only if $Q(R)$ is VNR and R_P is a valuation domain for every prime ideal $P \leqslant R$.*
(2) [11] *The* wk.gl.dim.$(R) \leqslant 1$ *if and only if R_P is a valuation domain for every prime ideal $P \leqslant R$.*
(3) [16] *R is an arithmetical ring if and only if R_P is a chained ring for every prime ideal $P \leqslant R$.*
(4) *R is a Gaussian ring if and only if R_P is a Gaussian ring for every prime ideal $P \leqslant R$.*
(5) *R is a Prüfer ring if and only if every 2-generated regular ideal is locally principal.*

It is worth noting that Prüfer conditions (1)–(4) are preserved under localization while condition (5) is not.

In our main theorem, it will be necessary to have the preservation of Prüfer condition (n) when passing to overrings. We will make use of two substantial results found in [2, Theorems 3.7 and 3.12].

Theorem 3.3. *Let R be a commutative ring.*

(1) *If R has Prüfer condition (n), then the total ring of quotients $Q(R)$ has Prüfer condition (n).*
(2) *The ring R has Prüfer condition (n) if and only if R is a Prüfer ring and $Q(R)$ has Prüfer condition (n).*

It is well known that every overring of a Prüfer ring is again a Prüfer ring (see for example [17, Chapter X]). Since an overring shares the same total ring of quotients as its "underring," Theorem 3.3 enables us to state the following:

Lemma 3.4. *Let R be a commutative ring with Prüfer condition (n). If T is an overring of R, then T has the same Prüfer condition (n).*

We now turn our attention to overrings of local commutative rings with Prüfer condition (n). We show that they have a particularly nice form. First we will need a lemma pointed out by Jim Coykendall. This result was shown in [21] for the case of local Gaussian rings.

Lemma 3.5. *If R is a local Prüfer ring, then the set $Z(R)$ of zero-divisors is a prime ideal.*

Proof. If this is not the case, then there exist two distinct prime ideals P and Q chosen maximally with respect to consisting only of zero-divisors by Zorn. Choose any $q \in Q - P$ and form the regular ideal (P, q). There now exists a regular element of the form $y = p + rq$, where $p \in P$ and $r \in R$. It follows that the 2-generated ideal (p, q) is regular so that, without harm, we have $(p, q) = (p)$ by Theorem 3.2(5). But then $p \mid y$ forcing the regularity of p. $\square$

We are now in a position to state and prove a crucial part of the main results.

Lemma 3.6. *If R is a local commutative ring with Prüfer condition (n) and if T is an overring of R, then T is a local ring with Prüfer condition (n). Moreover, $T = R_P$ for some prime ideal P of R.*

Proof. In light of Lemma 3.4, we need only show that $T = R_P$ for some prime ideal P of R. We begin by verifying that the result holds when R is a local Prüfer ring.

Set $S = \{s \in R - Z(R)\colon \frac{1}{s} \in T\}$. We show that $T = S^{-1}R$. The containment $T \supset S^{-1}R$ is straight forward. Choose any $t \in T$ and write $t = \frac{r}{s}$. Since R is a local Prüfer ring, Theorem 3.1 gives the ideal equality $(r, s) = (s)$ or $(r, s) = (r)$ in R. If $(r, s) = (r)$, then $r \mid s$ and $t = \frac{1}{d}$ for some $d \in R$, so that $t^{-1} \in R$. If $(r, s) = (s)$, then $t = c$ for some $c \in R$ so that $t \in R$. In either case, $T \subset S^{-1}R$.

Next, we show that $R - S$ is closed under scalar multiplication. Choose any $r \in R$ and $a \in R - S$. If $ar \notin R - S$, then $ar \in S$, so that $\frac{1}{ar} \in T$. Since $ar \in R - Z(R)$, a saturated multiplicatively closed set, we have $a \in R - Z(R)$. Thus, $\frac{1}{a} = r \cdot \frac{1}{ar} \in T$, so that $a \in S$, which is a contradiction.

Finally, we show that $R - S$ is closed under subtraction. If both r and s are zero-divisors, then $r - s \in Z(R)$ by Lemma 3.5 and hence, $r - s \in R - S$. If one of r or s is regular then it follows that $(r, s) = (r)$ (or (s)), so that $r - s = \alpha r$ for some $\alpha \in R$. Thus, $r - s \in R - S$ by the previous paragraph, so that $P = R - S$ is a prime ideal of R and $T = R_P$.

The remaining cases follow from Lemma 3.4, the previous remarks, and Theorem 3.3. □

4. Main results

In this section, we prove the main results of this article. We show that the five Prüfer conditions behave nicely in the regular conductor square (□). That is, we find necessary and sufficient conditions on A, T, and B in order that R has condition (n) for $n = 1, 2, 3, 4, 5$. A crucial step in the proof of the main result is to show that the local version holds.

Theorem 4.1. *Consider the regular conductor square* (□). *R is a local commutative ring with Prüfer condition (n) if and only if T is a local ring with Prüfer condition (n), A is a local Prüfer ring, and B is an overring of A.*

Proof. We begin with the Prüfer condition (5).

(⇒) Suppose that R is a local Prüfer ring. Then, by Lemmas 2.1(1) and 3.6, T is a local Prüfer ring. It is immediate that from the definitions and the regularity of C that A is a local Prüfer ring. Lemma 2.1(2) ensures that B is an overring of A.

(⇐) By Lemma 2.1(5), R is local. Choose any regular 2-generated ideal (r, s) in R. Since (r, s) is a regular ideal of the local Prüfer ring T, we may assume that $(r, s) = (r)$ in T. Thus, we have the equation $s = rt$ for some $t \in T$. Consider the image $\bar{t} \in B$. Since B is an overring of A we may write $\bar{t} = \frac{a}{b}$ with $a, b \in A$ and b regular. Since A is a local Prüfer ring and (a, b) is a regular ideal of A, we have $(a, b) = (a)$ or $(a, b) = (b)$. It follows that $\bar{t} \in A$ or $\bar{t}^{-1} \in A$. If $\bar{t} \in A$, then $t \in R$ and $(r, s) = (r)$ in R. If $\bar{t}^{-1} \in A$, then $\bar{t}$ is a unit in B and t is therefore a unit in the local ring T. That is, $t^{-1} \in T$ and $\overline{t^{-1}} = \bar{t}^{-1} \in A$ so that $t^{-1} \in R$. We now have the equation $st^{-1} = r$ and the ideal equation $(r, s) = (s)$ in R. The result follows.

The remaining three cases follow from Lemma 3.6, Lemma 2.1(2), the previous paragraph, and Theorem 3.3. □

Theorem 4.2. *Consider the regular conductor square* (□).

(1) *If R is a Prüfer ring, then A and T are Prüfer rings, and B_P is an overring of A_P for each prime* (*maximal*) *ideal P of R. Conversely, for each prime* (*maximal*) *ideal P of R, if A_P and T_P are Prüfer rings, and B_P is an overring of A_P, then R is a Prüfer ring.*
(2) *For $n = 1, 2, 3, 4$, R is a commutative ring with Prüfer condition (n) if and only if T has Prüfer condition (n), A_P is a Prüfer ring, and B_P is an overring of A_P for each prime* (*maximal*) *ideal P of R.*

Proof. (1) By Lemma 3.4, T is a Prüfer ring. It is immediate that A is a Prüfer ring. To see that B_P is an overring of A_P for each prime ideal $P \subset R$, we localize the conductor square (□) at P to obtain the diagram (□$_P$) displayed below

$$\begin{array}{ccc} R_P & \hookrightarrow & T_P \\ \downarrow & & \downarrow \\ A_P & \hookrightarrow & B_P. \end{array} \qquad (\Box_P)$$

Since R_P is a flat R-module, $(\Box_P)$ is also a regular conductor square. The regularity of C_P and Lemma 2.1(1) imply that T_P is an overring of R_P. It follows from Lemma 3.6 that T_P is a localization of R_P, so that B_P is an overring of A_P by Lemma 2.1(2).

For the converse, we show that the regular ideal (a,b) is locally principal. If $C \nsubseteq P$ then, by Lemma 2.1(4), there is a unique prime ideal $Q \subset T$ such that $R_P \simeq T_Q$. Since T is a Prüfer ring, $(a,b)T_Q$ is principal, and thus $(a,b)R_P$ is principal. If $C \subseteq P$, then we have the non-trivial conductor square $(\Box_P)$ with regular conductor C_P. Since A_P is a local Prüfer ring and B_P is an overring of A_P, by Lemma 3.6 we have that B_P is a local Prüfer ring. By Lemma 2.1(3), there is a one-to-one correspondence between the maximal ideals of T_P and the maximal ideals of B_P, so that T_P is local. But T_P is a Prüfer ring, and hence, a local Prüfer ring. We are now in the case of a conductor square $(\Box_P)$, in which T_P is a local Prüfer ring, A_P is a local Prüfer ring, C_P is regular, and B_P is an overring of A_P, so that, by Theorem 4.1, R_P is a local Prüfer ring. It now follows that $(a,b)R_P$ is principal at every prime ideal P of R.

(2) We verify the statement for the case when R is a semi-hereditary ring (Prüfer condition (1)). The proofs of the remaining cases are similar.

($\Rightarrow$) Since R is a semi-hereditary ring, T is a semi-hereditary ring by Lemmas 2.1(1) and 3.4. Since the homomorphic image of a valuation domain is a chained ring, one easily checks that A is an arithmetical ring. By Theorem 3.2(3) A_P is a chained ring and thus a Prüfer ring. The injection $R_P \hookrightarrow T_P \hookrightarrow Q(R_P)$ has been demonstrated.

($\Leftarrow$) Since T is a semi-hereditary ring, T_P is a Prüfer ring. Since A_P is a Prüfer ring and $R_P \hookrightarrow T_P \hookrightarrow Q(R_P)$, we have by (1) that R is a Prüfer ring. But then Lemma 2.1(1) and Theorem 3.3 ensure that R is a semi-hereditary ring. $\Box$

We can now give a complete characterization of Prüfer domains defined by means of a conductor square of the type $(\Box)$.

Corollary 4.3. *R is a Prüfer domain if and only if T is a Prüfer domain, A_P is a Prüfer ring, and B_P is an overring of A_P for each prime (maximal) ideal P of R.*

Acknowledgments

The author would like to thank North Dakota State University for their support during the research and writing of this article. The author is especially grateful to an unknown referee, Jim Coykendall, and Sarah Glaz for their helpful comments and suggestions. This paper is dedicated to the memory of James Brewer, my teacher and friend.

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